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Liouville quantum mechanics on a lattice from geometry of quantum Lorentz group

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Abstract. We consider the quantum Lobachevsky space $\mathbb{L}_q^3$, which is defined as a subalgebra of the Hopf algebra $\mathcal{A}_q(SL_2(\mathbb{C}))$. The Iwasawa decomposition of $\mathcal{A}_q(SL_2(\mathbb{C}))$ introduced by Podles and Woronowicz allows us to consider the quantum analogue of the horospheric coordinates on $\mathbb{L}_q^3$. The action of the Casimir element, which belongs to the dual to $\mathcal{A}_q$ quantum group $U_q(SL_2(\mathbb{C}))$, on some subspace in $\mathbb{L}_q^3$ in these coordinates leads to a second-order difference operator on the infinite one-dimensional lattice. In the continuous limit $q \to 1$ it is transformed into the Schrödinger Hamiltonian, which describes zero modes into the Liouville field theory (the Liouville quantum mechanics). We calculate the spectrum (Brillouin zones) and the eigenfunctions of this operator. They are q-continuous Hermite polynomials, which are particular cases of the Macdonald or Rogers–Askey–Ismail polynomials. The scattering in this problem corresponds to the scattering of the first two-level dressed excitations in the Z_N model in the very peculiar limit when the anisotropy parameter γ and $N \to \infty$, or, equivalently, $(\gamma, N) \to 0$.

1. Introduction

There are a lot of inter-relations between integrable one-dimensional systems of particles and integrable 2D field theories. One of them is the similarity in dynamics between solitons and the classical Calogero–Moser–Sutherland–Toda particles [1]. On a quantum level a similar phenomenon has been observed recently in [2, 3]‖. In particular, using results of [4], it was discovered there that the scattering of some special excitations in the Z_2-Baxter model coincides with the scattering, which is defined by asymptotics of the Macdonald polynomials for the root system A_1 [5]. They depend on two parameters and in the simplest case define the usual zonal spherical functions on reducible symmetric spaces of rank one.

Recall that the zonal spherical functions are defined as a normalized eigenfunctions of the Laplace–Beltrami operator on a symmetric space, which are invariant with respect to a stationary subgroup and have a free asymptotic. In other words, the Laplace–Beltrami operator in the spherical coordinates coincides up to a simple gauge transformation with the generalized Calogero–Moser–Sutherland potential [6]. The corresponding Jost function is the famous Harish–Chandra c-function [7], which is factorizable [8]. This fact is in agreement with the complete integrability of the Calogero–Mozer–Sutherland systems. In particular, the Jost functions for a symmetric non-compact space of rank one give rise to the scattering of a quantum particle on a semi-line on the Calogero–Moser potential $g^2/\sinh^2 x$.

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‖ Quantum dynamics was discussed partly in [1].

The same S-matrix arises in the scattering of two kinks in the XXX-model. In a similar fashion, the XXZ-model corresponds to the quantum symmetric space of rank one. But in this case the continuous semi-line is replaced on a semi-indefinite lattice.

To open quantum Toda Hamiltonian can be derived in a similar way [6, 9]. To this end the same Laplace–Beltrami operator is considered in the horospherical coordinates. If its eigenfunctions are independent on the horospherical 'angular' coordinates, then the Laplace–Beltrami operator is reduced to a second-order differential operator with constant coefficients. But if the functions have a non-trivial multiplier when their arguments are shifted along horospheres (more exactly, they belong to a representation, induced by a character of a nilpotent subgroup), then the Laplace–Beltrami operator acquires a non-trivial potential term, which is nothing other than the open Toda potential. The corresponding eigenfunctions are called the Whitteker functions [10, 11]. It was proved in [9] that the corresponding Jost functions are also factorizable. The simplest SL_2 case describes the scattering on the Liouville potential e^{-2x}.

We can consider this model as a simplified version of the ubiquitous Liouville field theory, for $\phi(\sigma, t)$ (the zero mode $x(t) = \phi(\sigma, t)|_{\sigma=0}$), which describes 2D induced gravity in the conformal gauge. Then e^{-2x} defines the circumference of a 1D 'universe'. The quantum mechanics of the zero modes sheds light on the spectrum of the full theory in the quasiclassical approach [12].

Our aim is to put this theory on a lattice. In other words, x will take discrete values. There are a few reasons for doing this. First of all, introducing the new parameter—the step of a lattice—is equivalent to the generalizations of corresponding spin chain Lagrangians. Next, this model in the 2D version modifies 2D gravity in a similar fashion to a finite-group approximation of gauge groups in the Yang–Mills theories. This modification of 2D gravity can be in principle useful in the quantization procedure. Finally, classical solutions of models on lattices may lead presumably to deformed tau functions connected with new integrable hierarchies, as well as to partition functions of some topological theories.

Here we consider the very simple quantum theory, which turns out to be exactly solvable. To derive the model we proceed to 'the second quantization' of the Liouville quantum mechanics using the formalism of quantum groups. Namely, we consider the quantum Lobachevsky space $\mathbb{L}_q^3$. It is defined as a subalgebra $\mathbb{L}_q^3$ of the Hopf algebra $\mathcal{A}_q(SL_2(\mathbb{C}))$, which is dual to the quantum Lorentz group $U_q(SL_2(\mathbb{C}))$. The algebra $\mathbb{L}_q^3$ is equipped with the right $U_q(SL_2(\mathbb{C}))$-module structure. In $\mathbb{L}_q^3$ exists an analogue of the horospherical coordinates, which are connected with the Iwasawa decomposition of $\mathcal{A}_q$ [13]. We calculate the Casimir element $\Omega_q \in U_q(SL_2(\mathbb{C}))$ in these coordinates. It is possible to reduce the operator to a subspace which is similar to the space of the induced representation of the nilpotent subgroup in the classical situation. The quantum horospherical variables are separated in this subspace. The operator Ω_q becomes a classical second-order difference operator on a one-dimensional lattice with the Liouville 'wall'. It differs slightly from relativistic Liouville, introduced in [14]. In the limit, when a step of the lattice vanishes, $\Omega_q|_{(q\to 1)}$ coincides with the Liouville Hamiltonian. The quantum Whitteker functions are the q-continuous Hermite polynomials, which are a particular case of the Rogers–Askey–Ismail polynomials [15], or the Macdonald polynomials for the A_1 root system [5]. Thus, the Macdonald polynomials in this case serve also to define the Whitteker functions, as well as the zonal spherical functions.

The Harish-Chandra c-function gives rise to the S-matrix, which coincides with the S-matrix for scattering of the first two levels in the Z_N-model [16] in the limit $N, \gamma \to \infty$, where γ is an anisotropy parameter, or, equivalently, $N, \gamma \to 0$.

2. Classical case

Let $\mathbb{L}^3 = SU_2\backslash SL_2(\mathbb{C})$ be is a homogeneous space of the second-order unimodular Hermitian positive definite matrices, which is a model of the classical Lobachevsky space. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1.$$

Then any $x \in \mathbb{L}^3$ can be represented as

$$x = g^\dagger g = \begin{pmatrix} \bar{\alpha}\alpha + \bar{\gamma}\gamma & \bar{\alpha}\beta + \bar{\gamma}\delta \\ \bar{\beta}\alpha + \bar{\delta}\gamma & \bar{\beta}\beta + \bar{\delta}\delta \end{pmatrix}. \tag{1}$$

The Iwasawa decomposition

$$g = kb \qquad g \in SL_2(\mathbb{C}) \qquad k \in SU_2 \qquad b \in AN - \text{Borel subgroup} \tag{2}$$

allows us to define the horospherical coordinates on $\mathbb{L}^3$. If

$$b = \begin{pmatrix} h & hz \\ 0 & h^{-1} \end{pmatrix}$$

then from (1)

$$x = b^\dagger b = \begin{pmatrix} \bar{h}h & \bar{h}hz \\ \bar{z}\bar{h}h & \bar{z}\bar{h}hz + (\bar{h}h)^{-1} \end{pmatrix}. \tag{3}$$

The triple $(H = \bar{h}h, z, \bar{z})$ is uniquely determined by x. It is called the horospherical coordinates of x. It follows from (1) and (3) that

$$H = \bar{\alpha}\alpha + \bar{\gamma}\gamma$$

$$H_z = \bar{\alpha}\beta + \bar{\gamma}\delta$$

$$\bar{z}H = \bar{\beta}\alpha + \bar{\delta}\gamma.$$

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

be the generators of the Lie algebra gl_2 and d_A, d_B, d_C and $d_D = -d_A$ be the corresponding Lie operators of the right shift on $\mathbb{L}^3$. In the horospherical coordinates they take the form

$$d_A = \tfrac{1}{2}H\partial_H - z\partial_z \qquad d_D = -d_A \qquad d_B = \partial_z \qquad d_C = Hz\partial_H - z^2\partial_z + H^{-2}\partial_{\bar{z}}. \tag{4}$$

The second Casimir

$$\Omega = d_A^2 + d_D^2 + d_B d_C + d_C d_B \tag{5}$$

in the horospherical coordinates takes the form

$$\Omega = \tfrac{1}{2}H^2\partial_H^2 + \tfrac{3}{2}H\partial_H + 2H^{-2}\partial_{z\bar{z}}^2. \tag{6}$$

Consider the eigenvalue problem

$$\tfrac{1}{2}\Omega\Phi_\lambda(H,z,\bar{z}) = -(\lambda^2+\tfrac{1}{4})\Phi_\lambda(H,z,\bar{z}) \tag{7}$$

and put

$$\Phi_\lambda(H,z,\bar{z}) = H^{-1}\exp \mathrm{i}\mu(z+\bar{z})\phi_\lambda(H)$$

where $\exp \mathrm{i}\mu(z+\bar{z})$ can be consider as a unitary character of the nilpotent subgroup N. Then for $x = \frac{1}{2}\log H$ and $\Psi_\lambda(x) = \phi_\lambda(H)$, equation (7) is transformed to

$$(-\tfrac{1}{4}d_{xx}^2 + 4\mu^2\mathrm{e}^{-4x})\Psi_\lambda(x) = 4\lambda^2\Psi_\lambda(x). \tag{8}$$

The solution to this equation

$$\Psi_\lambda(x) = k_{2\mathrm{i}\lambda}(2\mu\mathrm{e}^{-2x}) \tag{9}$$

is the Bessel–Macdonald function with the asymptotic behaviour

$$K_{2\mathrm{i}\lambda}(2\mu\mathrm{e}^{-2x}) \sim \frac{\pi}{\sin 2\pi\lambda}\left[\frac{(\mu\mathrm{e}^{2x})^{-2\mathrm{i}\lambda}}{\Gamma(1-2\mathrm{i}\lambda)} + \frac{(\mu\mathrm{e}^{2x})^{2\mathrm{i}\lambda}}{\Gamma(1+2\mathrm{i}\lambda)}\right]. \tag{10}$$

This solution is the so-called Whitteker function for $SL_2(\mathbb{C})$ [10, 11]. The two-body S-matrix

$$S(\lambda) = \frac{\Gamma(1+2\mathrm{i}\lambda)}{\Gamma(1-2\mathrm{i}\lambda)} \tag{11}$$

which is obtained from (10) describes the scattering of a quantum particle on the Liouville 'wall' e^{-4x}.

3. Quantum Lobachevsky space

Let $\mathcal{A}_q(SL_2(\mathbb{C}))$ $(0 < q \leqslant 1)$ be the algebra of functions on $SL_2(\mathbb{C})$ [13], which is defined as the factor algebra of the associate $\mathbb{C}$-algebra with generators $\alpha, \beta, \gamma, \delta$, with an anti-involution $*$: $\mathcal{A}_q \to \mathcal{A}_q$, $(ab)^* = b^*a^*$ and the following relations

$$\begin{gathered}
\alpha\beta = q\beta\alpha \qquad \alpha\gamma = \gamma\alpha \qquad \beta\delta = q\delta\beta \qquad \gamma\delta = q\delta\gamma \qquad \beta\gamma = \gamma\beta \\
\alpha\delta - q\beta\gamma = 1 \qquad \delta\alpha - q^{-1}\beta\gamma = 1 \qquad \beta\alpha^* = q^{-1}\alpha^*\beta + q^{-1}(1-q^2)\gamma^*\delta \\
\gamma\alpha^* = q\alpha^*\gamma \qquad \delta\alpha^* = \alpha^*\delta \qquad \gamma\beta^* = \beta^*\gamma \\
\delta\beta^* = q\beta^*\delta - q(1-q^2)\alpha^*\gamma \qquad \delta\gamma^* = q^{-1}\gamma^*\delta \\
\alpha\alpha^* = \alpha^*\alpha + (1-q^2)^2\gamma^*\gamma \qquad \beta\beta^* = \beta^*\beta + (1-q^2)(\delta^*\delta - \alpha^*\alpha) - (1-q^2)^2\gamma^*\gamma \\
\gamma^*\gamma = \gamma\gamma^* \qquad \delta\delta^* = \delta^*\delta - (1-q^2)\gamma^*\gamma.
\end{gathered} \tag{12}$$

The rest commutation relations can be read off from the rule $(ab)^* = b^*a^*$.

We cast the generators into the matrix form

$$w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \omega^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}.$$

With the comultiplication $\Delta : \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{A}_q$

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

the antipode $S : \mathcal{A}_q \to \mathcal{A}_q$

$$S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}$$

and the counit $\epsilon : \mathcal{A}_q \to \mathbb{C}$

$$\epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{A}_q$ becomes a Hopf algebra. In fact, it is a *-Hopf algebra, since

$$(\Delta(a))^* = \Delta(a^*)$$

and

$$S \circ * \circ S \circ * = \text{id}. \tag{13}$$

We define the *-Hopf subalgebra $\mathcal{A}_q(SU_2)$ by the generators

$$\mathcal{A}_q(SU_2) = \left\{ \omega_c = \begin{pmatrix} \alpha_c & -q\gamma_c^* \\ \gamma_c & \alpha_c^* \end{pmatrix} \right\} \tag{14}$$

and the relations

$$\alpha_c^*\alpha_c + \gamma_c^*\gamma_c = 1 \qquad \alpha_c\alpha_c^* + q^2\gamma_c^*\gamma_c = 1$$

$$\gamma_c^*\gamma_c = \gamma_c\gamma_c^* \qquad \alpha_c\gamma_c^* = q\gamma_c^*\alpha_c \qquad \alpha_c\gamma_c = q\gamma_c\alpha_c.$$

Then

$$\omega_c^*\omega_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

In a similar way

$$\mathcal{A}_q(AN_q) = \left\{ \omega_d = \begin{pmatrix} h & n \\ 0 & h^{-1} \end{pmatrix} \right\} \tag{16}$$

$$hh^* = h^*h \qquad hn = qnh \qquad nh^* = q^{-1}h^*n$$

$$nn^* = q^2n^*n + (1-q^2)((h^*h)^{-2} - 1).$$

The Iwasawa decomposition in the quantum context takes the form [13]

$$\omega = \omega_c\omega_d \qquad \omega \in \mathcal{A}_q(SL_2(\mathbb{C})) \qquad \omega_c \in \mathcal{A}_q(SU_2) \qquad \omega_d \in \mathcal{A}_q(AN_q). \tag{17}$$

Natural description of commutation relations (12) can be obtained from the construction of the quantum double. It was implemented in [17], where $\mathcal{A}_q(SL_2(\mathbb{C}))$ is described as a special quantum double of $\mathcal{A}_q(SU_2)$, and (13) is derived by means of the corresponding R-matrix.

The quantum Lobachevsky space $\mathbb{L}_q^3$ is a *-subalgebra of $\mathcal{A}_q(SL_2(\mathbb{C}))$ generated by the bilinear constituents

$$\omega^*\omega = \begin{pmatrix} \alpha^*\alpha + \gamma^*\gamma & \alpha^*\beta + \gamma^*\delta \\ \beta^*\alpha + \delta^*\gamma & \beta^*\beta + \delta^*\delta \end{pmatrix} = \begin{pmatrix} p & s \\ s^* & r \end{pmatrix}. \tag{18}$$

Evidently, * acts as

$$p^* = p \qquad (s)^* = s^* \qquad r^* = r.$$

We don't need the explicit form of the commutation relations between p, s, s^* and r—they can be derived from (12).

Introduce a new generator z instead of n

$$n = hz.$$

Then due to (15), (16) and (18)

$$p = H = h^*h = hh^* \qquad s = Hz \qquad s^* = z^*H \qquad r = z^*Hz + H^{-1}. \tag{19}$$

The triple (H, z, z^*) generates the horospherical coordinates in the algebra $\mathbb{L}_q^3$. It follows from (12) that

$$\begin{aligned} &Hz = q^2zH \qquad H^{-1}z = q^{-2}zH^{-1} \qquad z^*H = q^2Hz^* \qquad z^*H^{-1} = q^{-2}H^{-1}z^* \\ &zz^* = q^2z^*z + (q^2-1)(1-H^{-2}). \end{aligned} \tag{20}$$

Consider now the complex associative algebra $U_q(SL_2)(\mathbb{C})$ with unit 1, generators A, B, C, D and the relations

$$\begin{aligned} &AD = DA = 1 \qquad AB = qBA \qquad BD = qDB \\ &AC = q^{-1}CA \qquad CD = q^{-1}DC \\ &[B, C] = \frac{1}{q-q^{-1}}(A^2 - D^2). \end{aligned} \tag{21}$$

In fact, it is the Hopf algebra, where

$$\begin{aligned} &\Delta(A) = A \otimes A \qquad \Delta(D) = D \otimes D \\ &\Delta(B) = A \otimes B + B \otimes D \qquad \Delta(C) = A \otimes C + C \otimes D \end{aligned} \tag{22}$$

$$\begin{aligned} &\epsilon\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &S\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix}. \end{aligned} \tag{23}$$

There exists a non-degenerate bilinear form $\langle u, a\rangle$: $U_q \times \mathcal{A}_q \to C$ such that

$$\langle \Delta(u) a \otimes b\rangle = \langle u, ab\rangle \qquad \langle u \otimes v, \Delta(a)\rangle = \langle uv, a\rangle$$

$$\langle 1_U, a\rangle = \epsilon_A(a) \qquad \langle u, 1_A\rangle = \epsilon_U(u) \qquad \langle S(u), a\rangle = \langle u, S(a)\rangle.$$

It takes the form of the generators

$$\left\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right\rangle = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \qquad \left\langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right\rangle = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$$
$$\left\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \left\langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

Moreover, $U_q(SL_2(\mathbb{C}))$ is *-Hopf algebra in duality, where the involution is defined by the pairing

$$\langle u^*, a\rangle = \overline{\langle u, (S(a))^*\rangle}. \tag{25}$$

The element

$$\Omega_q := \frac{(q^{-1}+q)(A^2+D^2)-4}{2(q^{-1}-q)^2} + \tfrac{1}{2}(BC+CB) \tag{26}$$

is a Casimir element, since it commutes with any $u \in U_q(SL_2(\mathbb{C}))$.

The right action of $u \in U_q(SL_2(\mathbb{C}))$ on $\mathcal{A}_q$ is defined as [4]

$$a \cdot u := (u \otimes \mathrm{id})(\Delta(a)). \tag{27}$$

It is the algebra action:

$$a \cdot (uv) = (a \cdot u) \cdot v \tag{28}$$

which satisfies the 'Leibnitz rule'

$$(ab) \cdot u = \sum_j (a \cdot u_j^1)(b \cdot u_j^2) \tag{29}$$

where $\Delta(u) = \sum_j u_j^1 \otimes u_j^2$. The left action is defined in the similar way.

The right action on the generators takes the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot A = \begin{pmatrix} q^{1/2}\alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & q^{-1/2}\delta \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot B = \begin{pmatrix} 0 & \alpha \\ 0 & \gamma \end{pmatrix}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot C = \begin{pmatrix} \beta & \\ \delta & 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot D = \begin{pmatrix} q^{-1/2}\alpha & q^{1/2}\beta \\ q^{1/2}\gamma & q^{1/2}\delta \end{pmatrix}. \tag{30}$$

We will define now the right action of $U_q(SL_2(\mathbb{C}))$ on $\mathbb{L}_q^3$, which endows the latter with the structure of the right *-module. For any $a \in \mathbb{L}_q^3$ define the normal ordering using (12)

$$: a :: = \sum_k c_k a_{1,k}^* a_{2,k} \tag{31}$$

where $a^*_{1,k}$ ($a_{2,k}$) are monoms depending on $\alpha^*, \beta^*, \gamma^*, \delta^*$ ($\alpha, \beta, \gamma, \delta$). Then the right action on $\mathbb{L}^3_q$, which will be denoted as $(a) \cdot u$, is defined as follows

$$(a) \cdot u = \sum_k c_k a^*_{1,k}(a_{2,k} \cdot u). \tag{32}$$

In particular, from (30)

$$(p, s, s^*, r) \cdot A = (p, 0, s^*, 0) \qquad (p, s, s^*, r) \cdot B = (o, p, 0, s^*)$$
$$(p, s, s^*, r) \cdot C = (s, 0, r, 0) \qquad (p, s, s^*, r) \cdot D = (0, s, 0, r).$$

To obtain the action of generators of $U_q(SL_2(\mathbb{C}))$ on the horospheric coordinates it is necessary to express them through $\alpha^*, \beta^*, \gamma^*, \delta^*$ and $\alpha, \beta, \gamma, \delta$ ordering them in correspondence with (31), and then applying them to (32). Two expressions have the normal order from the very beginning (see (18) and (19))

$$H = \alpha^*\alpha + \gamma^*\gamma \qquad Hz = \alpha^*\beta + \gamma^*\delta. \tag{33}$$

For H^{-1} we can write a representation as a formal series

$$H^{-1} = (\alpha^*)^{-1} \sum_{k=0}^{\infty} (-1)^k q^{-2k}((\alpha^*)^{-1}\gamma^*)k(\gamma\alpha^{-1})^k\alpha^{-1}. \tag{34}$$

It can be proved by induction that

$$\begin{aligned} H^l &= (\alpha^*)^l \sum_{k=0}^{l} q^{(l-1)k} \frac{[l]_{q^2}!}{[k]_{q^2}![l-k]_{q^2}!} y^{*k} y^k \alpha^l \qquad l \geqslant 0 \qquad y = \gamma\alpha^{-1} \\ H^{-l} &= (\alpha^*)^{-l} \sum_{k=0}^{\infty} (-1)^k q^{-(l+1)k} \frac{[l+k-1]_{q^2}!}{[k]_{q^2}![l-k]_{q^2}!} y^{*k} y^k \alpha^{-l} \qquad l > 0. \end{aligned} \tag{35}$$

Here we use the standard notations

$$[k]_{q^2} = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sinh k\hbar_2}{\sinh \hbar_2} \tag{36}$$

where $q = \exp(-\hbar_2)$†, and $[k]_{q^2}! = [k]_{q^2}[k-1]_{q^2} \ldots 1$.

The normal order for z and z^* can be derived from (33) and (34)

$$z = \xi + \kappa \qquad \xi = \alpha^{-1}\beta \tag{37}$$

and

$$\kappa = \gamma^* H^{-1} \alpha^{-1} = \sum_{k=0}^{\infty} (-1)^k q^{-2k} (y^*)^{k+1} y^k \alpha^{-2} \qquad y = \gamma\alpha^{-1}. \tag{38}$$

† The subscript 2 is used here to distinguish this constant from the usual Planck constant $\hbar_1$, which is present from the very beginning in front of the derivatives in the classical group approach.

Also

$$\xi\kappa^k = q^{2k}\kappa^k\xi + (q^{2k} - 1)\kappa^{k+1}$$

and therefore

$$z^n = (\xi + \kappa)^n = \sum_{l=0}^{n} q^{l(n-1)} \frac{[n]_{q^2}!}{[l]_{q^2}![n-l]_{q^2}!} \kappa^l \xi^{n-1}. \tag{39}$$

From (37)

$$\begin{aligned}
&(z) \cdot A = q^{-1}z \qquad (z^*) \cdot A = z^* \\
&(z) \cdot B = q^{-1/2} \qquad (z^*) \cdot B = 0 \\
&(z) \cdot C = -q^{1/2}z^2 \qquad (z^*) \cdot C = q^{-1/2}H^{-2} \\
&(z) \cdot D = qz \qquad (z^*) \cdot D = z^*.
\end{aligned} \tag{40}$$

In the same way, from (35) for any $r \in \mathbb{Z}$,

$$\begin{aligned}
&(H^r) \cdot A = q^{r/2}H^r \qquad (H^r) \cdot B = 0 \\
&(H^r) \cdot C = q^{(1-r)/2}[r]_{q^2}H^r z \\
&(H^r) \cdot D = q^{-r/2}H^r.
\end{aligned} \tag{41}$$

Formally, we can consider this relation for any $r \in \mathbb{C}$, since H (35) can be defined as a positive definite Hermitian operator.

The relations (40) and (41) demonstrate the action of $U_q(SL_2(\mathbb{C}))$ on the horospherical coordinates of $\mathbb{L}_q^3$. Moreover, using the same notations for this representation as for the generators of $U_q(SL_2(\mathbb{C}))$, we can prove that in the classical limit equations (40) and (41) are transformed in the Lie derivatives (4)

$$\begin{aligned}
&d_A = \lim_{q\to 1} \partial_q A \qquad d_D = \lim_{q\to 1} \partial_q D \\
&d_B = \lim_{q\to 1} B \qquad d_C = \lim_{q\to 1} C.
\end{aligned}$$

As for the quantum Casimir (26), it is easy to check that

$$\lim_{q\to 1} \Omega_q = \tfrac{1}{2}\Omega + \tfrac{1}{4} \tag{42}$$

(Ω is the classical Casimir (6)), as it has to be.

4. Eigenfunctions of the quantum Casimir and scattering in spin chains

The eigenvalue problem

$$(f(H, z, z^*)) \cdot \Omega_q = -\lambda^2 f(H, z, z^*) \tag{43}$$

where $f(H, z, z^*)$ is a Laurent series in H, z, z^*, can in principle be written down explicitly by means of the relations from the previous section and the rules (28) and (29).

In the similar fashion to the classical case, we choose a special form of $f(H, z, z^*)$. Let

$$e_q^v = \sum_{n=0}^{\infty} \frac{v^n}{(q;q)_n}$$

where

$$(a;q)_n \begin{cases} 1 & n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & n>0 \end{cases}$$

is the q-exponent [15]. Then we assume

$$f(H, z, z^*) = e_{q^2}^{i\mu q^{-1/2}(1-q^2)z^*} F_\lambda(H) e_{q^2}^{i\mu q^{-3/2}(1-q^2)z}$$

and $F_\lambda(H) = \sum_r a_{r+1}(\lambda) H^r$. Thus

$$f(H, z, z^*) = \sum_{m=0}^{\infty} \sum_r \sum_{n=0}^{\infty} \frac{(i\mu)^{m+n} q^{-m-n}(1-q^2)^{m+n}}{(q^2,q^2)_m (q^2,q^2)_n} a_r(\lambda) X^{*m} H^r X^n \quad (44)$$

where $X = q^{1/2} z$.

From (39)

$$v^{(m,r,n)} = X^{*m} H^r X^n$$
$$= q^{(m+n/2)} \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{q^{k(m-1)+l(n-1)} [m]_{q^2}! [n]_{q^2}!}{[k]_{q^2}! [m-k]_{q^2}! [l]_{q^2}! [n-l]_{q^2}!} \xi^{*m-k} \kappa^{*k} H^r \kappa^l \xi^{n-l}. \quad (45)$$

As follows from (12),

$$\kappa^{*k} = (\alpha\gamma)^{*-k} \kappa^k (\alpha\gamma)^k.$$

It can be found from (29) and (30) for $n \geqslant 0$ that

$$(\xi^n)\cdot A = q^{-n}\xi^n \qquad (\xi^n)\cdot B = q^{-1/2}[n]_{q^2}\xi^{n-1}$$
$$(\xi^n)\cdot C = -q^{1/2}[n]_{q^2}\xi^{n+1} \qquad (\xi^n)\cdot D = q^n\xi^n.$$

For $n \geqslant 0$, after some algebra, we obtain

$$(\kappa^n)\cdot A = q^{-n}\kappa^n \qquad (\kappa^n)\cdot B = 0$$
$$(\kappa^n)\cdot C = -q^{n+\frac{1}{2}}[2n]_{q^2}\kappa^n\xi - q^{2n+\frac{1}{2}}[n]_{q^2}\kappa^{n+1} \qquad (\kappa^n)\cdot D = q^n\kappa^n.$$

Then the following relation

$$(\alpha\gamma)^{-1} = \kappa^{*-1} H^{-2} + \kappa$$

allows us to derive the actions of the generators on the monom $\omega^{(k,r,l)} = \kappa^{*k} H^r \kappa^l$

$$(\omega^{(k,r,l)}) \cdot A = q^{r/2-l}\omega^{(k,r,l)}$$

$$(\omega^{(k,r,l)}) \cdot B = 0$$

$$(\omega^{(k,r,l)}) \cdot C = q^{-k+l+(r+l)/2}[k]_{q^2}\omega^{(k-1,r-2,l)} - q^{l-(r-1)/2}[2l-r]_{q^2}\omega^{(k,r,l)}\xi - q^{2l-(r-l)/2}[l-r]_{q^2}\omega^{(k,r,l+1)}$$

$$(\omega^{(k,r,l)}) \cdot D = q^{-r/2+1}\omega^{(k,r,l)}.$$

In agreement with (45)

$$(v^{(m,r,n)}) \cdot A = q^{(r/2)-n}v^{(m,r,n)}$$

$$(v^{(m,r,n)}) \cdot B = q^{(r-1)/2}[n]_{q^2}v^{(m,r,n-1)}$$

$$(v^{(m,r,n)}) \cdot C = q^{m+n+(r-2)/2}[m]_{q^2}v^{(m-1,r-2,n)} - q^{-r/2}[n-r]_{q^2}v^{(m,r,n+1)}$$

$$(v^{(m,r,n)}) \cdot D = q^{-(r/2)+n}v^{(m,r,n)}.$$

Using (28) and (29), we obtain

$$(v^{(m,r,n)}) \cdot \Omega_q = \left[\frac{r+1}{2}\right]_{q^2}^2 v^{(m,r,n)} + q^{m+n+r-2}[m]_{q^2}[n]_{q^2}v^{(m-1,r-2,n-1)}.$$

Substituting it in (44), we come to

$$(f(H,z,z^*)) \cdot \Omega_q = \mathrm{e}_{q^2}^{\mathrm{i}\mu q^{-1/2}(1-q^2)z^*}\sum_r a_r \left[\frac{r+1}{2}\right]_{q^2}^2 H^r \mathrm{e}_{q^2}^{\mathrm{i}\mu q^{-1/2}(1-q^2)z} - \mu^2 \mathrm{e}_{q^2}^{\mathrm{i}\mu q^{-3/2}(1-q^2)z^*} \times \sum_r a_r q^{r-2} H^{r-2} \mathrm{e}_{q^2}^{\mathrm{i}\mu q^{-3/2}(1-q^2)z}.$$

Therefore, (43) is reduced to

$$\sum_r a_r \left[\frac{r+1}{2}\right]_{q^2}^2 H^r - \mu^2 \sum_r a_r q^{r-2} H^{r-2} = -\lambda^2 \sum_r a_r H^r.$$

This relation is equivalent to

$$\frac{qF_\lambda(qH) - 2F_\lambda(H) + q^{-1}F_\lambda(q^{-1}H)}{(q-q^{-1})^2} - \mu^2 q^{-2} H^{-2} F_\lambda(qH) = -\lambda^2 F_\lambda(H).$$

After the substitution

$$\mathcal{F}(H) = HF(H)$$

we come to our main equation

$$\frac{\mathcal{F}_\lambda(qH) - 2\mathcal{F}_\lambda(H) + \mathcal{F}_\lambda(q^{-1}H)}{(q-q^{-1})^2} - \mu^2 q^{-3} H^{-2} \mathcal{F}_\lambda(qH) = -\lambda^2 \mathcal{F}_\lambda(H). \quad (46)$$

It is the lattice counterpart of the Liouville equation (8)—in the limit $q \to 1$, ($\hbar_2 \to 0$) it is transformed in (8). Here the second derivative is replaced on the second difference operator. This naive modification of (8) can be predicted easily from the very beginning. The less obvious modification is the shift of the argument in the potential operator. It will of course be crucial for solving the eigenvalue problem exactly.

Using the shift operator in the form

$$\exp(\hbar_2 \partial)\mathcal{F}(H) = \mathcal{F}(qH)$$

we can rewrite the last relation as

$$\left[\frac{(\sinh)^2\left(\frac{\hbar_2\partial}{2}\right)}{(\sinh)^2\hbar_2} - \mu^2 q^{-3} H^{-2} e^{\hbar_2\partial}\right]\mathcal{F}_\lambda(H) = -\lambda^2 \mathcal{F}_\lambda(H). \tag{47}$$

To solve (46) assume that

$$\mu^2 q^{-3}(q - q^{-1}) = 1 \tag{48}$$

and replace the eigenvalue parameter

$$\lambda = [\mathrm{i}\theta]_{q^2} = \frac{\sin\left(\frac{\hbar_2\theta}{2}\right)}{\sinh \hbar_2} \qquad x = \cos \hbar_2\theta. \tag{49}$$

Since (46) is the difference operator, put

$$H = q^{-n} \tag{50}$$

$$c_n(x) = \frac{1}{(q^2, q^2)_n}\mathcal{F}_\lambda(H)\,|_{H=q^{-n}}\,. \tag{51}$$

In the new variables equation (46) takes the form

$$(1 - q^{2n+2})c_{n+1}(x) + c_{n-1}(x) = 2xc_n(x). \tag{52}$$

This equation has the form of the recurrence relation for orthogonal polynomials. If we put $c_{-1} = 0$ and $c_0 = 1$, then (51) defines the q-continuous Hermite polynomials, which are a particular case of the Rogers–Askey–Ismail polynomials [15], or, equivalently, the Macdonald polynomials for the A_1 root system [5]. Namely,

$$c_n(\cos\hbar_2\theta) = C_n(\cos\hbar_2\theta; t \mid q^2)\,|_{t=0} \tag{53}$$

where $C_n(\cos\hbar_2\theta; t|q^2)$ is the Rogers–Askey–Ismail polynomial

$$C_n(\cos\hbar_2\theta; t \mid q^2) = \sum_{a+b=n, a,b\in\mathbb{Z}^+} \frac{(t;q^2)_a (t;q^2)_b}{(q^2;q^2)_a (q^2;q^2)_b} e^{\mathrm{i}\hbar_2\theta(a-b)}$$

and for the Macdonald polynomials P_n

$$\begin{aligned} |\mathcal{F}_\lambda(H)|_{H=q^{-n}} &= (q^2;q^2)_n c_n(\cos\hbar_2\theta) \\ &= (t;q^2)_n P_n(e^{\mathrm{i}\hbar_2\theta}; t \mid q^2)\,|_{t=0}\,. \end{aligned} \tag{54}$$

It is worth noting that the degrees of polynomials n now play the role of the 'space' variable (51), while their arguments are related to the eigenvalue parameter λ (49), (50). Note that the argument θ, which defines the energy $E = \lambda^2/2$ by (49), lies in the interval $\theta \in [0, 4\pi/\hbar_2)$. A similar phenomenon also arises for the zonal spherical functions on quantum 'non-compact' symmetric spaces (see [2, 3], where the case of $U_q(SU(1,1))$ was considered).

The asymptotic behaviour of c_n for $n \to \infty$ is the following [15]

$$(q^2; q^2)_n c_n(\cos\hbar_2\theta) \sim \left\{ \frac{q^{-\mathrm{i}n\theta}}{(q^{2\mathrm{i}\theta}; q^2)_\infty} + \frac{q^{\mathrm{i}n\theta}}{(q^{-2\mathrm{i}\theta}; q^2)_\infty} \right\}. \tag{55}$$

Thus, the Harish–Chandra c-function is equal to

$$c(\lambda) = \frac{1}{(q^{-2\mathrm{i}\theta}; q^2)_\infty}.$$

To find a spin chain, which exhibits the same scattering, it is convenient to consider the extended theory and switch on the additional parameter t (53), (54). This theory was considered in [3]. It was demonstrated there that the Harish–Chandra c-functions for the generic A_1 Macdonald polynomials give rise for the scalar S-matrix to the scattering of two special excitations in the Z_N-model [16] in the $N \to \infty$ regime. It has two parameters—the modular parameter τ and the anisotropy parameter γ. In fact, τ is a modular parameter of an elliptic curve and γ is a point on it. They are parameters of the Sklyanin algebra [18–20], which allows a solution of the corresponding Yang–Baxter equation. The parameters of the Macdonald polynomials are related to the latter as follows:

$$q = \mathrm{e}^{2\mathrm{i}\pi\tau} \qquad t = q^{\mathrm{i}\gamma/\pi\tau}. \tag{56}$$

Some interesting solutions corresponding to points in the space (τ, γ) were considered in [3]. In our case, $t = 0$ (54). Since $\mathrm{Im}\tau > 0$ and $\mathrm{Re}\tau = 0$, then $\gamma \to \infty$. Simultaneously, $N \to \infty$.

As follows from relation (5.2) of [3], there is duality in the scattering picture in this model

$$\left(N, \frac{-\mathrm{i}\pi\tau}{\gamma}\right) \sim \left(\frac{-\mathrm{i}\pi\tau}{\gamma}, N\right).$$

Thus the same S-matrix can be realized in the Z_N-Baxter model in the regime

$$\gamma \to 0 \qquad N \to 0.$$

5. Conclusion

We have found that the wavefunctions of the q-Liouville Hamiltonian lie in the same family of Macdonald polynomials as the wavefunctions of the q-Sutherland Hamiltonian [2–4]. In the former case $t = 0$, and in the latter $t = q$ (see (54)). The Macdonald polynomials have an interpretation as the 'zonal spherical functions' on the Sklyanin quantum algebra [3]. Thus there exists the transition from the usual $q = 1$ Sutherland Hamiltonian to the Liouville quantum mechanics through the Sklyanin quantum algebra. It is natural to conjecture that this transition can be generalized on the arbitrary number of particles. Recall that both Hamiltonians can also be considered as two special reductions of the Casimir on $U_q(SL_2)$, $q \leqslant 1$. But this unification is performed in a different way by increasing the degrees of freedom.

One of the ways to study quantum integrable systems is to put the space variable on a lattice. *In particular, for the Liouville field theory it was done firstly in* [21], at least quasiclassically. Our approach is different—instead of the discretization of the space variable we discretized the zero modes of the Liouville field. It is similar to the substitution a continuous gauge group on a discrete subgroup in the Monte Carlo simulations of gauge theories. Introducing the lattice is equivalent to introducing an infinite hard wall instead of the Liouville exponential potential—the wavefunctions vanish for $n \leqslant 0$. The theory, as a modified version of the zero-mode dynamics of 2D gravity, is still ultraviolet-free since for small distances the potential vanishes $H^{-2} = q^{2n}_{n\to+\infty} = 0$ (see (50), (55)).

On the other hand the system under consideration resembles the Ruijsenaars relativistic Toda model [14] for two particles, though it does not coincide with it. In spite of the latter it allows us to solve it exactly. It is possible also to find the explicit solution in the classical case after the substitution $\mathrm{i}\partial \to p$. Unfortunately, this solution does not have a clear group-theoretical interpretation, like the solutions of the open non-relativistic Toda model [6]. At the same time the Ruijsenaars solutions are natural q-deformations of the non-relativistic Toda solutions. It will be interesting to find a similar interpretation for the solution coming from the quantum Lorentz group.

Starting from the Hamiltonian (46) it is easy to guess the form of the lattice N-body Toda quantum mechanics. But it will not be easy to justify this Hamiltonian by means of the Iwasawa decomposition of $\mathcal{A}_q(SL(N,\mathbb{C}))$ [17] using the same type calculations as above. Nevertheless, investigations of q-Whitteker functions in a general case and their relations with representations of quantum groups is plausible.

The next desirable generalization of this model is its two-dimensional version in the quantum and classical form. Note that one of the possible versions of the q-deformed classical Liouville field theory can be reconstructed in principle, as in the classical theory, from the q-deformed WZW model, proposed, for example, in [22].

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